HIGH-DIMENSIONAL REPRESENTATIONS OF GROUPS IN MUSIC THEORY

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Abstract. We present the link between the representation of degree one of a musical object of n-cardinality modeled through a cyclic group and the roots of unity. We exemplify the representations of the group of transpositions and inversions and we restate the analogy with a matrix polynomial of cyclotomic form whose roots are representations of the musical object in higher degrees. Then the degree one representation of the T-group is compared to the JQZ group in a Klumpenhouwer network.
Keywords: Cyclotomic polynomials, transformational groups, representations of finite groups, T/I group, Klumpenhouwer networks.

1. Introduction

Consider an abstract musical object represented by a set of cardinality n. Then, every rhythmic or harmonic musical entity can be associated with a monic polynomial $T^n - 1$ according to its cardinality. Take the linear factor decomposition of the polynomial

$$T^n - 1 = (T-1)(T-\zeta_n)(T-\zeta_n^2)(T-\zeta_n^3)\cdots(T-\zeta_n^{n-1}) = \prod_{k=0}^{n-1}(T-\zeta_n^k).$$

Similarly, a musical chromatic scale is represented by

$$T^{12} - 1 = (T-C)(T-C\sharp)(T-D)(T-E\flat)\cdots(T-B) = \prod_{k=C}^{B}(T-\zeta_{12}^k). \tag{1}$$

Likewise, the assignment for a diatonic scale is determined by music theory. Then each note in the $A\flat$ locrian scale in Figure 1 is described as roots of unity:

$$T^7 - 1 = (T-A\flat)(T-B\flat\flat)(T-C\flat)(T-E\flat)\cdots(T-B) = \prod_{k=A\flat}^{G\flat}(T-\zeta_7^k). \tag{2}$$

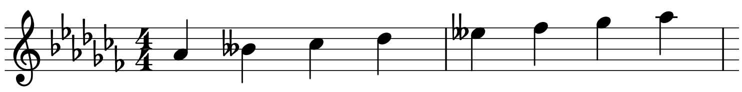

Figure 1. Locrian mode on $A\flat$.

It can be naturally generalized to harmonic or rhythmic universes of n-microtones or n-pulses. For example, the degree of the extension of the rational field for the chromatic scale of the unity with primitive root $e^{2\pi i/12}$ is $[\mathbb{Q}(\zeta):\mathbb{Q}] = 4$. In a harmonic universe, primitive roots generate what is musically called the circle of fifths, fourths, and the total chromatic scale.

Thus, musical elements can be modeled by notions of cyclotomic field theory. On the other hand, cyclotomic polynomials of degree $\phi(n)$ have been studied in relation to the theory of rhythmic canons and tessellations, introducing Galois theory in the mathematical music theory [FR05, Fid07, Gil07, Cau13].

2. N-dimensional spheres and T/I group representations

Within the paradigm of the actions of a specific group on a musical group; An advantage that is sought when defining new analytical groups isomorphic to some type of mathematical group is to achieve a conceptual economy of the transformations between musical objects. In this way, the new groups reveal a pattern not visible throughout the musical work.

On the other hand, the matrix representations of transformational groups such as PLR or $UTTs$ have already been discussed previously. For example, the group $\mathcal{J}$ [FN18] improves the uniformity of the type of operation PLR between musical chords together with the action of some element of the symmetric group S_3. As [FN18, Section 1.1, p. 283] the observing analyst chooses the most suitable group for analysis under certain musical premises.

Following the same argument explained, we propose only to tempt the possible representations of cyclic groups and the relationship that may exist between them with regard to musical analysis. We will consider first the $n - spheres$ that are presented in the analogy from the cyclic group of integers modulo n and are part of the theory of representation of finite groups. The representation of degree $\delta = 1$ for the cyclic group that models a chromatic scale is given through homomorphism

$$\begin{aligned} \Psi : \mathbb{Z}/12\mathbb{Z} &\longrightarrow \mathbb{C}^*, \\ k &\longmapsto e^{\frac{2\pi ik}{12}}. \end{aligned} \tag{3}$$

Then, the musical transposition T_n behaves geometrically like the rotation of a point in the one-dimensional sphere $\mathbb{S}^1$ and is reduced to complex multiplication in the group of roots of the unity $(\zeta_n, \cdot)$. At this point, the DFT [Ami16] can be seen within a system of operations in a representation of groups of degree $\delta = 1$ of the musical objects.

Let $\mathbb{M}$ be the subset of major and minor chords of the power set of the chromatic scale. For a fixed number $k \in \mathbb{Z}/12\mathbb{Z}$ if the assignment of the notes follows the more conventional clockwise direction with the musical representation of the chromatic circle, any major chord X can be written

$$X = \{e^{2\pi ik/12}, e^{2\pi ik/12} \cdot e^{-2\pi 4/12}, e^{2\pi ik/12} \cdot e^{-2\pi i7/12}\}.$$

And in general, any kind of chord or scale $\{X, Y, \ldots\}$ can be written similarly. The transposition by $m \in \mathbb{Z}$ semitones to each note of the major chord is formulated as

$$\begin{aligned} T_{\zeta_n^m} : \mathbb{M} &\longrightarrow \mathbb{M}, \\ X &\longmapsto X \cdot \zeta_n^m. \end{aligned} \tag{4}$$

The inversion function for a particular note in any harmonic universe is

$$I_{\zeta_n^m}(e^{\frac{2\pi ik}{n}}) \mapsto e^{\frac{2\pi i(-k)}{n}} \cdot e^{\frac{2\pi im}{n}};$$

so it can be expressed as the complex conjugation followed by a multiplication by transposition for every chord or scale:

$$\begin{aligned} I_{\zeta_n^m} : \mathbb{M} &\longrightarrow \mathbb{M}, \\ X &\longmapsto -X \cdot \zeta_n^m. \end{aligned} \tag{5}$$

Now consider a representation Ψ of degree $\delta = 2$ of the cyclic group that sends it to the general linear group of non-singular matrices with complex elements

$$\begin{aligned} \Psi : \mathbb{Z}/12\mathbb{Z} &\longrightarrow GL_2(\mathbb{C}), \\ k &\longmapsto \begin{bmatrix} e^{\frac{2\pi ik}{12}} & 0 \\ 0 & e^{\frac{-2\pi ik}{12}} \end{bmatrix}. \end{aligned} \tag{6}$$

Representing the transposition by two semitones through the matrix representation $T_{[2]}^{\mathbb{C}}$ from the note F to G in this vector space of square matrices is

$$\begin{bmatrix} e^{\frac{2\pi i2}{12}} & 0 \\ 0 & e^{\frac{-2\pi i2}{12}} \end{bmatrix} \begin{bmatrix} e^{\frac{2\pi i5}{12}} & 0 \\ 0 & e^{\frac{-2\pi i5}{12}} \end{bmatrix} = \begin{bmatrix} e^{\frac{7\pi i}{6}} & 0 \\ 0 & e^{\frac{-7\pi i}{6}} \end{bmatrix}.$$

Returning to the polynomial, a musical object such as the chromatic scale can be represented

$$T^n - I_2 = (T - \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]) \left(T - \begin{bmatrix} e^{\frac{2\pi i}{n}} & 0 \\ 0 & e^{\frac{-2\pi i}{n}} \end{bmatrix}\right) \left(T - \begin{bmatrix} e^{\frac{2\pi i 2}{n}} & 0 \\ 0 & e^{\frac{-2\pi i 2}{n}} \end{bmatrix}\right) \cdots \left(T - \begin{bmatrix} e^{\frac{2\pi i(n-1)}{n}} & 0 \\ 0 & e^{\frac{-2\pi i(n-1)}{n}} \end{bmatrix}\right) = \prod_{k=0}^{n-1} \left(T - \begin{bmatrix} e^{\frac{2\pi i k}{n}} & 0 \\ 0 & e^{\frac{-2\pi i k}{n}} \end{bmatrix}\right), \tag{7}$$

where I_2 is the identity matrix.

3. Ring model of musical representations

The equation (7) opens up the possibility of expanding the representation to degree $\delta = m$ in the general linear group of any musical abstract object with cardinality n through the following polynomial expression whose roots are representations of some degree:

$$T^n - I_m = \prod_{k=0}^{n-1} \left(T - \begin{bmatrix} \sharp_{11}^k & \cdots & \sharp_{1m}^k \\ \vdots & \Psi : \mathbb{Z}/n\mathbb{Z} \longrightarrow GL_m(\mathbb{V}) & \vdots \\ \sharp_{m1}^k & \cdots & \sharp_{mm}^k \end{bmatrix}\right). \tag{8}$$

The symbol $\sharp$ in the equation (8) represents an element of scalars that can be complex or of another field. With this definition, a representation of degree $\delta = m$ can be associated to each point of the classes of tones or rhythms as a matrix root that is attached to the field of rationals in analogy with $\mathbb{Q}[\zeta]$.

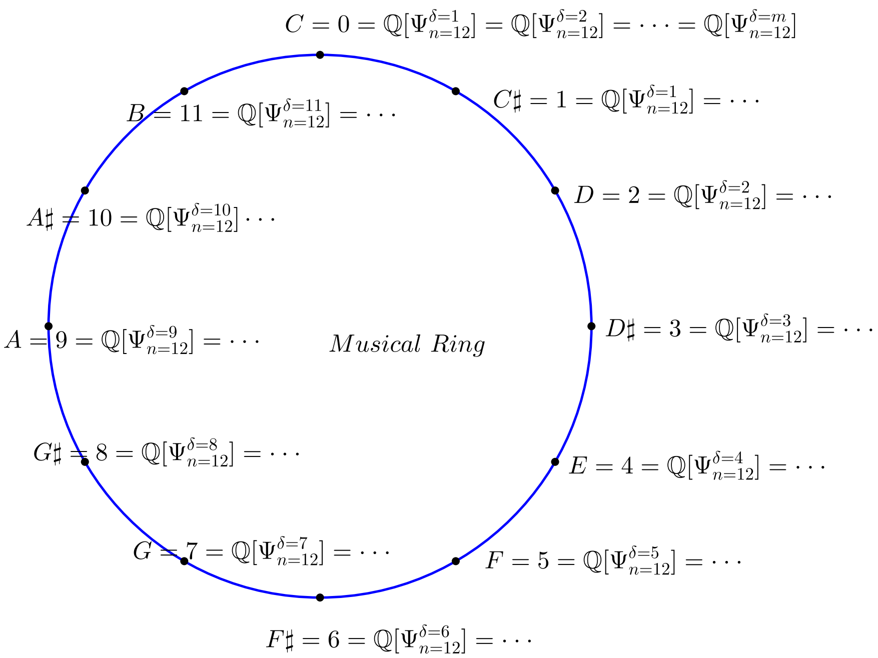

Figure 2. Ring of possible degree representations.

Similar representations can be seen in Figure 2, where the universe has cardinality $n = 12$ and the degree to which each note can be represented ranges from $1 \leq \delta \leq m$. Transpositions T_n and inversions I_n are induced from the type of representation in which each note or rhythm is addressed. It should be clarified that the equality between extensions arises in the sense of the musical operation, i.e, a transposition by two semitones is T_2 or also if $\delta = 2$ in the matrix multiplication of the equation (7); since two equivalent representations need a linear transformation between both vector spaces.

4. Arrows in musical representations

The different degrees of representation can be equipped in Klumpenhouwer networks [Lew90] just like the T/I group and the non-contextual group JQZ [Jed19], where $J = I_7$, $Q = I_{11}$ and $Z = I_4$.

$$
\begin{tikzcd}
G \arrow[dd, "{T_7,\ T_{exp\left(\frac{2\pi i7}{12}\right)},\ (ZQ)^{-1}}"'] \arrow[dr, "{T_3,\ T_{exp\left(\frac{2\pi i3}{12},\right)},\ JZ}"] & \\
& B\flat \arrow[dl, "{T_4,\ T_{exp\left(\frac{2\pi i4}{12}\right)},\ QJ}"] \\
D &
\end{tikzcd}
$$

Figure 3. The G minor chord under different simultaneous musical transformations: transposition of the group T/I, transposition in the representation of degree $\delta = 1$ and transposition by means of the non-contextual group JQZ.

Figure 3 illustrates the conjunction of three groups representing the internal structure of the chord by transposition. A basic entity such as a solo chord is insufficient to choose the ideal group to represent its structure, so the criteria for choosing which group to use could be reduced to the most economical in purely notational or intuitive terms: T/I.

5. Conclusion and discussion

It has been shown that the representation of cyclic groups can emerge analogically through the roots of unity and extend into the manipulation of square matrices in high dimensions. Currently, a more general approach has been developed where the groups exemplified above become a particular case within the category theory and are therefore considered within a category that includes all types of groups of musical actions [PAE18]. Thus, each of the degrees of representations would be part of this category of group tools of transformational analysis. However, this short article may also point to the fundamentally musical analytical need to opt for a representation of degree $\delta = 1$ or one of degree $\delta = m$. Then, the questions are oriented to find out the utility of representations of a higher degree. For example, if the DFT space is a representation of degree one:

- What analytical benefits are there in representations of higher degrees?
- What about the coefficients and phases of subsets of $\mathbb{Z}/n\mathbb{Z}$?
- Does it increase the amount of musical information that can be extracted with the usual operations in higher dimensions?

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